Concurrent topology optimization of structures and their composite microstructures

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A B S T R A C T
Different from the independent design of macrostructures or material microstructures, a two-scale topology optimization algorithm is proposed by using the bi-directional evolutionary structural optimization (BESO) method for the concurrent design of the macrostructure and its composite microstructure. It is assumed that the macrostructure is made of composite materials whose effective properties are calculated through the homogenization method. By conducting finite element analysis of both structures and materials, sensitivity numbers at the macro- and micro-scale levels are derived. Then, the BESO method is used to iteratively update the macrostructures and the composite microstructures according to the elemental sensitivity numbers at both scales. Some 2D and 3D numerical examples are presented to demonstrate the effectiveness of the proposed optimization algorithm. A variety of optimal macrostructures and optimal material microstructures have been obtained.

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1. Introduction

Structural optimization is becoming increasingly important due to the limited material resources, environmental impact and technological competition, all of which demand lightweight, low-cost and high-performance structures. Structural topology optimization technique seeks to achieve the best performance of a structure while satisfying various constraints such as a given amount of material. Compared with size and shape optimizations, topology optimization provides much more freedom and allows the designer to create novel and highly efficient conceptual designs for structures. Over the last two decades, various topology optimization algorithms, e.g. homogenization method [1], solid isotropic material with penalization (SIMP) [2–4], evolutionary structural optimization (ESO) [5,6], and level-set technique [7,8] have been developed. Unlike the continuous density-based topology methods, the ESO/BESO methods represent the structural topology and shape with discrete design variables (solid or void) so that the resulting design gives a clear structural boundary [6,9]. ESO was originally developed based upon a simple concept of gradually removing redundant or inefficient material from a structure so that the resulting topology evolves towards an optimum [6]. A later version of the ESO method, namely the bi-directional ESO (BESO) method, allows not only removing materials, but also adding materials to the design domain [10,11]. It has been demonstrated that the current BESO method is capable of generating reliable and practical topologies for various types of structures with high computational efficiency [9,12].

Currently, topology optimization techniques are mainly used to solve one-scale design problems either for the optimal design of macrostructures to improve their structural performance or for the material design to develop new microstructures with prescribed or extreme properties [13–17]. The optimal design of a macrostructure assumes that the structure is composed of given materials which can be selected from the available material database. The material design assumes that the material is made of periodic base cells and the macroscopic effective properties of the heterogeneous material are homogenized (averaged) according to the microstructure of the base cell. The inverse problem is a typical topology optimization problem for the material design which seeks an optimal microstructure of the base cell with prescribed or extreme macroscopic properties [18]. Such a microstructural design approach greatly enriches available materials which can be selected for constructing macrostructures by virtue of material properties, so as to satisfy required performance specifications.

The material selection for macrostructures is a complex process which involves not only material properties but also the service conditions such as structure shape, applied loadings and boundary conditions etc. Instead of selecting materials, Huang et al. [19] formulated a two-scale topology optimization problem and directly designed the microstructures of cellular materials and composites for given shapes of macrostructures using the BESO method. Zhang
and Sun [20] discussed the scale effect in two-scale topology optimization of cellular materials and structures. However, an ideal design of a macrostructure should be the structure which has an optimal macroscopic topology, and meanwhile, is composed of materials/composites with optimal microstructures. That is, we should concurrently design the topologies of a macrostructure and its material microstructure. Inspired by biological systems, Rodrigues et al. [21] proposed a hierarchical computational procedure by integrating the macrostructures with a series of local material microstructures using the continuum density-based method, and Coelho et al. [22] extended this hierarchical procedure to the three-dimensional elastic structures. However, it is impossible to find optimal microstructure point-to-point even using parallel computing techniques. By constraining the volume fractions at the macro-level and the micro-level separately, Liu et al. [23] conducted a concurrent topology optimization of materials and structures where the macrostructure is solely composed of a material. The method was extended by Yan et al. [24] for minimizing the compliance of 2D thermoelastic structure, and by Niu et al. [25] for maximizing structural fundamental frequency. Deng et al. [26] studied the multi-objective design of lightweight thermoelastic structures using the concurrent optimization technique to minimize the structural compliance and the thermal expansion of a certain surface simultaneously. Unfortunately, the continuum density-based method cannot absolutely preclude “grey areas” with intermediate densities in the structural topology. The material properties at “grey areas” are roughly estimated through the material interpolation scheme but their microstructures are still unknown.

This paper proposes a two-scale topology optimization approach based on the BESO method for concurrently designing structures and materials. Different from the continuum density-based method, BESO utilizing discrete design variables is more suitable for concurrent topology optimization of structures and materials because there is no need to assume any properties or microstructures for intermediate materials for finite element analysis. The layout of the paper is as follows. A two-scale concurrent optimization model is established and illustrated in Section 2. The homogenization of effective material properties and sensitivity analysis of both macrostructures and materials are presented in Section 3. The procedure for implementing the BESO method for the concurrent optimization of macrostructures and material microstructure is given in Section 4. Section 5 presents several 2D and 3D numerical examples to demonstrate the effectiveness of the proposed optimization algorithm. Concluding remarks are given in Section 6.

2. Concurrent optimization model

Consider a macrostructure with known boundary conditions and external forces as illustrated by Fig. 1(a). The macrostructure is composed of two-phase composite with microstructures (Fig. 1(b)) periodically repeated by the base cell (Fig. 1(c)). In Fig. 1(c), phase 1 with density $\rho_1$ and phase 2 with density $\rho_2$ are represented by green and grey respectively. It is assumed that phase 1 is stiffer and heavier than phase 2 ($E_1 > E_2$, $\rho_1 > \rho_2$). The optimization objective is to find the spatial optimal topologies for both the macrostructure and its material microstructure so that the resulting macrostructure has the best loading-carrying capability for a given total weight. Optimizations at the two scales will be integrated into one system and resolved concurrently. For such a two-scale optimization problem, there are two finite element models, namely the macro model for macrostructure and the micro model for the base cell of material. To seek the maximum stiffness (or minimum mean compliance) of the macrostructure, the concurrent topology optimization can be formulated as

Find $x_i, x_j$ \hspace{1cm} (i = 1, 2, ..., M;  \ j = 1, 2, ..., N)

Minimize $C(x_i, x_j) = \frac{1}{2} \sum_{i=1}^{M} U_i^T K(x_i, x_j) U_i$ \hspace{1cm} (1)

Subject to $K(x_i, x_j) U = F$ \hspace{1cm} (2a)

$m(x_i, x_j) - W_j m_0 = 0$ \hspace{1cm} (2b)

$x_i, x_j = 0 \ or \ 1$ \hspace{1cm} (2c)

where $C$ denotes the mean compliance of the structure, $F$ and $U$ represent the external force vector and the nodal displacement vector of the structural at the macro level, respectively. $K$ is the stiffness matrix of the macrostructure which can be assembled by the elemental stiffness matrix $K_i$. $M$ is the total number of finite elements in the macro structure. $W_j$ is the prescribed weight fraction of the final design. $x_i$ and $x_j$ are the binary design variables for the macro and micro models, respectively. In the macro model, $x_i = 1$ represents a solid element (two-phase composite or uniform material) and $x_i = 0$ represents a void element. In the micro model, when an element is made of phase 1, $x_j = 1$ and when phase 2, $x_j = 0$. $m_0 = \sum_{j=1}^{N} V_j \rho_1$ is the reference weight of the structure when the whole design domain is fully filled with phase 1. The weight of the design, $m$, can be expressed by

$m = \sum_{i=1}^{M} x_i V_i \rho_i, (x_i = 0 \ or \ 1)$ \hspace{1cm} (3)

where $\rho_i$ is the density of a solid element in the macro model. It is related to the micro model through mass conservation as

$N \sum_{j=1}^{N} \frac{V_j \rho_1 + (1 - x_j) \rho_2}{V_i}, (x_j = 0 \ or \ 1)$ \hspace{1cm} (4)
where \( N \) is the total number of elements in the micro model and \( V_j \) is the volume of element \( j \) in the micro model.

It can be seen that the objective function in Eq. (1) depends on not only the design variable \( x_i \) in the macro model but also the design variable \( x_j \) in the micro model. It should also note that the weight constraint can be satisfied by removing the elements in the macrostructure or switching phase 1 to phase 2 in the base cell. Therefore, this is a typical two-scale topology optimization problem where the topologies of the macrostructure and material base cell should be determined simultaneously.

To obtain clear topologies, the material interpolation scheme is adopted at both scales. At the micro scale, the elemental elasticity matrix can be expressed as \[ D^{\text{el}} = x_i^p D^{\text{el}}_1 + (1 - x_i^p) D^{\text{el}}_2 \] (5)

where \( p \) is the exponent of penalization and \( p = 3 \) is used throughout this paper. \( D^{\text{el}}_1 \) and \( D^{\text{el}}_2 \) denote the elasticity matrices for phase 1 and phase 2 respectively. For a cellular material, phase 2 is void with \( D^{\text{el}}_2 = 0 \).

At the macro scale, the elemental elasticity matrix can be expressed as \[ D^{\text{MA}} = x_i^p D^{\text{el}}_1 \] (6)

where \( D^{\text{el}}_1 \) is the effective elasticity matrix which should be computed from the micro-scale analysis of the material base cell through the classical homogenization method [27,28].

The rejection/addition criterion of the BESO method is based on the sensitivity numbers. Sensitivity numbers can be obtained from derivative-free parameters such as elemental von Mises stress or from gradient information which is derived through sensitivity analysis [9–11]. The finite element analysis and sensitivity analysis for concurrent design of structures and materials will be given in the following section.

3. Finite element analysis and sensitivity analysis

The finite element analysis should be conducted for both the structure at the macro-scale and the material base cell at the micro-scale. The static behaviour of the macrostructure can be represented by the equilibrium Eq. (2a). The stiffness matrix \( K \) can be assembled by the elemental stiffness matrix, \( K_i \)

\[ K = \int_Y b^T D^{\text{MA}} b dY \] (8)

where \( b \) is the strain–displacement matrix.

When the size of the material base cell is very small compared with the size of the structure body, the homogenization theory can be applied to obtain the effective elasticity matrix of the material. In the micro FE model, the material base cell is analyzed by imposing the periodic boundary conditions [29] as

\[ k = \int_Y b^T D^{\text{el}} b dY \] (9)

where \( k = \int_Y b^T D^{\text{el}} b dY \) is the stiffness matrix of the microstructure of the base cell, \( b \) is the strain–displacement matrix at the micro-scale level. The right hand side of Eq. (8) denotes the external forces caused by uniform strain fields e.g., \( [1, 0, 0]^T, [0, 1, 0]^T, [0, 0, 1]^T \) for 2D cases. \( u \) denotes the displacement field of the base cell caused by these uniform strain fields. As a result, the homogenized elasticity matrix can be expressed by

\[ D^\text{MA} = \frac{1}{|Y|} \int_Y D^{\text{el}} (I - bu) dY \] (10)

where \( I \) is an identity and \( |Y| \) is the total area or volume of the material base cell.

To implement topology optimization techniques such as the BESO method, sensitivity analysis is necessary for guiding the search direction of the optimization algorithm. At the macro-scale level, the mean compliance depends on the material properties in each element. With the help of Eqs. (1), (2), (6) and (7), the derivation of the mean compliance with respect to the design variable \( x_i \) can be expressed using the adjoint variable method [30], as

\[
\frac{\partial C}{\partial x_i} = -\frac{1}{2} U^T \frac{\partial K}{\partial x_i} U - \frac{1}{2} \rho x_i \int_{V_i} \mathbf{B}^T D^{\text{el}} b dV, \mathbf{U},
\]

(11)

where \( U \) is the displacement vector of the \( i \)th element in the macrostructure. Similarly, the derivation of total weight \( m \) of the structure with respect to \( x_i \) can be calculated by using Eqs. (3) and (4) as

\[
\frac{\partial m}{\partial x_i} = \sum_{j=1}^{N} V_j [x_j \rho_1 + (1 - x_j) \rho_2]
\]

(12)

In the proposed optimization framework, it is assumed that the structure is composed of a sole composite or cellular material. Therefore, the design variable, \( x_i \) in the material base cell is related to all solid elements in the structure. The sensitivity of the mean compliance against \( x_i \) is equal to the summation of the derivatives of the mean compliance for all elements in the macrostructure as

\[
\frac{\partial C}{\partial x_i} = -\frac{1}{2} \sum_{j=1}^{M} x_j \frac{\partial K}{\partial x_i} U_i = -\frac{1}{2} \sum_{j=1}^{M} x_j U_i^T \int_{V_i} \mathbf{B}^T \frac{\partial D^{\text{el}}}{\partial x_i} b dV, U_i
\]

(13)

Similarly, the derivation of total weight \( m \) with respect to \( x_i \) can be expressed as

\[
\frac{\partial m}{\partial x_i} = \sum_{j=1}^{M} x_j \frac{\partial p}{\partial x_i} V_j = \sum_{j=1}^{M} x_j V_j [\rho_1 - \rho_2]
\]

(14)

At the micro-scale level, the derivation of \( D^{\text{el}} \) with respect to \( x_i \) can be obtained according to the adjoint variable method, as

\[
\frac{\partial D^{\text{el}}}{\partial x_i} = \frac{1}{|Y|} \int_Y (I - bu)^T \frac{\partial D^{\text{el}}}{\partial x_i} (I - bu) dY
\]

(15)

With the help of the material interpolation scheme in Eq. (5), the above equation can be rewritten as

\[
\frac{\partial D^{\text{el}}}{\partial x_i} = \frac{p x_i^{p-1}}{|Y|} \int_Y (I - bu)^T (D_{1} - D_{2}) (I - bu) dY
\]

(16)

From Eq. (16), it can be seen that the sensitivities of elements in the material base cell is also related to the displacement field of the structure. In other words, the structure in turn affects the design of materials.

4. Numerical implementation and BESO procedure

In the BESO method, the sensitivity numbers which denote the relative ranking of the elemental sensitivities will be used to update the design variables \( x_i \) and \( x_j \). For a multi-scale problem, elemental sensitivities at different scales should be normalized by the variation of their weights so that they can be compared and ranked at the same level. The sensitivity number of the \( i \)th element in the macrostructure for minimizing the mean
compliance under a prescribed structure weight constraint can be defined as

$$x_i = -2 \frac{\partial C}{\partial \delta m_i} \left/ \frac{\partial m}{\partial \delta x_i} \right.$$  \hspace{1cm} (17)

Similarly, the sensitivity number of the jth element in the micro material base cell can be defined as

$$x_j = -2 \frac{\partial C}{\partial \delta m_j} \left/ \frac{\partial m}{\partial \delta x_j} \right.$$  \hspace{1cm} (18)

Using Eqs. (10), (11), (13), and (16), the sensitivity number $x_i$ at the macro level and sensitivity number $x_j$ at the micro level can be easily calculated. To realize the true concurrent designs of structure and material, sensitivity numbers $x_i$ and $x_j$ are sorted together from the highest to the lowest. Because the design variables $x_i$ and $x_j$ are restricted to be discrete values either 0 or 1, we can devise a simple scheme for updating the design variables $x_i = 0$ and $x_j = 0$ for elements with the lowest sensitivity numbers and $x_i = 1$ and $x_j = 1$ for elements with highest sensitivity numbers.

Numerical instabilities such as checkerboard pattern and mesh-dependency problem are common phenomenon in the topology optimization techniques based on the finite element analysis [31]. Here, a mesh-independent filter for discrete design variables [10] is applied for the ith elemental sensitivity number at the macro level as

$$x_i = \frac{\sum_{r=r_i}^{M} w(r) x_i}{\sum_{r=r_i}^{M} w(r)} \hspace{1cm} (19)$$

where $r_i$ denotes the distance between the centers of elements $j$ and $r$ in the macrostructure. $w(r)$ is the weight factor given as

$$w(r) = \begin{cases} r_{\text{min}} - r & \text{for } r_{\text{min}} < r < r_{\text{min}} \\ 0 & \text{for } r \geq r_{\text{min}} \end{cases} \hspace{1cm} (20)$$

where $r_{\text{min}}$ is the filter radius which can be specified by the user. The jth elemental sensitivity number at the micro level can also be filtered in the same way.

Due to the discrete design variables used in the BESO algorithm, Huang and Xie [10] proposed that the elemental sensitivity number should be further modified by averaging with its historical information to improve the convergence of the solution. Thus, the sensitivity number after the first iteration can be further modified by

$$x_i = \frac{1}{2} \left( x_{i,k} + x_{i,k+1} \right) \hspace{1cm} (21)$$

where $k$ is the current iteration number. Then replace $x_{i,k}$ with $x_i$ for the next iteration.

The whole BESO procedure for concurrently designing a macrostructure and its material microstructure is outlined as follow:

**Step 1:** Define BESO parameters such as the target weight fraction $W_f$ of the whole structure $W_f$, the evolutionary ratio $ER$ (normally $ER=0.2$) and the filter radii $r_{\text{min}}^{\text{macro}}$ for the macro level and $r_{\text{min}}^{\text{micro}}$ for the micro level. Construct initial designs of the macrostructure and the material base cell.

**Step 2:** Carry out finite element analysis for the material base cell.

**Step 3:** Calculated the homogenization material properties $D^H$ according to Eq. (5).

**Step 4:** Construct the finite element model for the macrostructure and substitute the resulting material properties $D^H$ into the model. Then carry out finite element analysis for the structure at the macro level.

**Step 5:** Calculated the elemental sensitivity numbers $x_i$ at the macro level according Eq. (17) and $x_j$ at the micro level according Eq. (18).

**Step 6:** Filter elemental sensitivity numbers $x_i$ and $x_j$ using Eq. (19), respectively, and then average them with their historical information using Eq. (21).

**Step 7:** Determine the target weight fraction of the whole structure for the next design. When the current weight fraction $W_{f,k}$ is larger than the target weight fraction $W_f$, reduce the weight fraction as

$$W_{f,k+1} = W_{f,k}(1 - ER) \hspace{1cm} (22)$$

If the resulting $W_{f,k+1}$ is less than $W_f$, then $W_{f,k+1}$ is set to $W_f$. Similarly, the weight fraction should be increased when $W_{f,k}$ is less than the target weight fraction $W_f$

$$W_{f,k+1} = W_{f,k}(1 + ER) \hspace{1cm} (23)$$

If the resulting $W_{f,k+1}$ is larger than $W_f$, then $W_{f,k+1}$ is set to $W_f$.

**Step 8:** According to the relative ranking of the elemental sensitivity numbers at the macro level and the micro level, assign the design variables $x_i$ to 1 (solid) and $x_j$ to 1 (phase 1) for elements with highest sensitivity numbers, and assign $x_i$ to 0 (void) and $x_j$ to 0 (phase 2) for elements with lowest sensitivity numbers so that the resulting weight fraction of the whole structure is equal to $W_{f,k+1}$. The process can be similar to the BESO method in the references [10,12]. As the result, the topologies of the macrostructure and the material base cell are updated simultaneously.

**Step 9:** Repeated Step 2–8 until both the weight constraint of the whole structure is satisfied and the objective function is convergent.

5. Numerical examples and discussions

In this section, we present several examples for concurrently designing macrostructures and their composite microstructures. The sizes and loads for all examples are chosen to be dimensionless. It is also assumed that phase 1 has Young’s modulus $E_1 = 2.0$, Poisson’s ratio $\mu_1 = 0.3$ and density $\rho_1 = 8$, and phase 2 has Poisson’s ratio $\mu_2 = 0.3$ and density $\rho_2 = 1$ in the two-phase composite. Young’s modulus of the phase 2, $E_2$, is chosen to be different values for the examples in order to investigate the effect of the phase contrast for two-phase composites. The base cell which represents the macrostructure of the materials is discretized into $51 \times 51$ 4-node quadrilateral elements for the 2D examples and $25 \times 25 \times 25$ 8-node brick elements for the 3D examples, respectively. For simplicity, a uniform meshes with element size $1 \times 1$ and $1 \times 1 \times 1$ are assigned to all 2D and 3D macrostructures, respectively. The used BESO parameters are evolution rate $ER = 2\%$ and filter radii $r_{\text{min}}^{\text{macro}} = 3$ for macrostructures and $r_{\text{min}}^{\text{micro}} = 2$ for microstructures of materials.

5.1. 2D cantilever

Fig. 2 shows a cantilever with a concentrated vertical loading at the center of the right edge and length $L = 40$, height $H = 100$. Young’s modulus of the phase 2 is assumed to be $E_2 = 1.5$.

When the prescribed weight fraction $W_f = 25\%$, Fig. 3 shows the resulting macrostructure and its material microstructure (green for phase 1 and grey for phase 2). Obviously, the resulting composite is orthotropic with higher stiffness in $y$-direction (vertical direction) than that in $x$-direction (horizontal direction).

Fig. 4 plots its evolution histories of the objective function (mean compliance), the weight fraction and the topologies of the macrostructure and its material microstructure. It can be seen that elements in the macrostructure are removed gradually first and then added back while elements in the material base cell are switched from stiff material (phase 1) to soft material (phase 2).
step by step. This process illustrates that the designs of the macrostructure and its material are interactive with each other and the topologies of the macrostructure and the material base cell will be gradually changed by the proposed BESO method.

Table 1 lists the values of the objective function (mean compliance), the optimized macrostructures and base cells for the cantilever with $L = 40$ and $H = 100$ under various weight fractions. It can be seen from Table 1 that the stiffer and heavier phase 1 in the composite base cell becomes less and less when the weight constraint $W_f/C_3$ decreases. Table 1 also indicates that optimized topologies of macrostructures have no significant difference for the cases with the weight constraint $W_f/C_3$ to be larger than 12.5%.

When the weight constraint $W_f/C_3$ is equal to 12.5%, the composite base cell is almost composed of phase 2 only. Therefore, the optimized topology of the macrostructure has to be changed significantly when the weight constraint $W_f/C_3$ is less than 12.5%. To demonstrate the advantage of the proposed two-scale optimization, we optimized the structure by assuming that the material is composed of phase 1 only. The resulting mean compliances and optimized macrostructures from such a one-scale optimization are also listed and compared in Table 1. It can be found that the two-scale optimized macrostructure has smaller mean compliance than that from the one-scale optimization under the same weight fraction. This is reasonable because the design degrees of freedom of the two-scale optimization problem are much more than that of the one-scale optimization problem.

Table 2 lists the resulting two-scale topologies and the effective elasticity matrices under various weight constraints for a cantilever with $L = 40$ and $H = 40$. As expected, the dimension of the structure greatly affects the optimized material microstructure. In the cases, the stiffness in $x$-direction is higher than that in $y$-direction. Compared with the results for the short cantilever listed in Table 1, it can be seen that the long cantilever should be composed of the material with high stiffness in $x$-direction so as to improve its capacity to resist bending. This is consistent with general principles in structural analysis. From Tables 1 and 2, it is seen that the mean compliance increases monotonically with the reduction

![Fig. 2. A cantilever with length $L$ and height $H$.](image)

![Fig. 3. Optimized macrostructure and its material microstructure for the cantilever with weight fraction $W_f=25\%$ and $E_2=1.5$: (a) topology of the cantilever; (b) topology of the two-phase composite base cell (green for phase 1 and grey for phase 2); (c) $2 \times 2$ unit cells; (d) effective elasticity matrix $D^H$.](image)

![Fig. 4. Evolution histories of the mean compliance, weight fraction and two-scale topologies when $E_2=1.5$ and weight fraction $W_f=25\%$.](image)
Obviously, the microstructures of the cantilevers are totally anisotropic which is different from that of the macrostructures and their material microstructures when the weight fraction is low (12.5\%). Additionally, it is interesting to note that the resulting materials are all orthotropic and their topologies are symmetric about \( x \) and \( y \) axes although there is no such symmetric constraints imposed by the optimization algorithm.

### 5.2. 2D beam

In this example, we will conduct the two-scale optimal designs for a 2D beam as shown in Fig. 5. Due to symmetry, only left half of the beam is considered for the analysis of the macrostructure.

Table 3 shows the resulting topologies of the macrostructures and the material microstructures for the beam with various weight constraints when \( E_2 = 1.5 \). Obviously, the microstructures of the beam are totally anisotropic which is different from that of the above cantilevers. Table 4 shows the resulting topologies of the macrostructures and their material microstructures when the weight fraction is low (12.5\%). It can be found that with the same weight fraction \( W_f = 30\% \), there is no significant difference for the optimized macrostructures in Table 3 and Table 4, but the topologies of the material base cells are totally different. It is reasonable that the

<table>
<thead>
<tr>
<th>Weight fraction ( W_f/% )</th>
<th>Two-scale Objective ( C )</th>
<th>Macro structure</th>
<th>Micro base cell</th>
<th>Effective elasticity matrix ( D^H )</th>
<th>One-scale (phase 1 only) Objective ( C )</th>
<th>Macro structure</th>
</tr>
</thead>
</table>
| 70                          | 24.28                       |                |                | \[
|                              |                             | \[2.0303 \ 0.6055 \ 0.0 \\
|                              |                             | \ [0.6055 \ 2.0201 \ 0.0 \\
|                              |                             | \ [0.0 \ 0.0 \ 0.7060 \\
| 50                          | 26.03                       |                |                | \[
|                              |                             | \ [1.8953 \ 0.5595 \ 0.0 \\
|                              |                             | \ [0.5595 \ 1.8640 \ 0.0 \\
|                              |                             | \ [0.0 \ 0.0 \ 0.6528 \\
| 25                          | 28.36                       |                |                | \[
|                              |                             | \ [1.7308 \ 0.5143 \ 0.0 \\
|                              |                             | \ [0.5143 \ 1.7142 \ 0.0 \\
|                              |                             | \ [0.0 \ 0.0 \ 0.6000 \\
| 12.5                        | 29.59                       |                |                | \[
|                              |                             | \ [1.6499 \ 0.4950 \ 0.0 \\
|                              |                             | \ [0.4950 \ 1.6499 \ 0.0 \\
|                              |                             | \ [0.0 \ 0.0 \ 0.5775 \\
| 5                           | 58.89                       |                |                | \[
|                              |                             | \ [1.6484 \ 0.4945 \ 0.0 \\
|                              |                             | \ [0.4945 \ 1.6484 \ 0.0 \\
|                              |                             | \ [0.0 \ 0.0 \ 0.5769 \\

of the weight fraction. Additionally, it is interesting to note that the resulting materials are all orthotropic and their topologies are symmetric about \( x \) and \( y \) axes although there is no such symmetric constraints imposed by the optimization algorithm.
Table 3
The optimized macrostructures and their composite microstructures for the 2D beam with various weight constraints when $E_2 = 1.5$.

<table>
<thead>
<tr>
<th>Weight fraction $W_f$ (%)</th>
<th>Objective C</th>
<th>Macro structure (left half)</th>
<th>Micro base cell</th>
<th>Effective elasticity matrix $D^{ij}$</th>
</tr>
</thead>
</table>
| 70                       | 17.02       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
| 50                       | 18.28       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
| 30                       | 19.68       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
| 10                       | 21.78       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[

Table 4
The optimized macrostructures and their composite microstructures for the 2D beam with various weight constraints when $E_2 = 1.2$.

<table>
<thead>
<tr>
<th>Weight fraction $W_f$ (%)</th>
<th>Objective C</th>
<th>Macro structure (left half)</th>
<th>Micro base cell</th>
<th>Effective elasticity matrix $D^{ij}$</th>
</tr>
</thead>
</table>
| 45                       | 20.94       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
| 40                       | 21.66       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
| 30                       | 23.14       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
| 10                       | 27.22       | ![Macro structure image]     | ![Micro base cell image] | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[
|                          |             |                             |                 | \[

Fig. 6. A 3D cantilever under a point load with length $L$, width $B$ and height $H$.

constituents of the composite are distributed in a different manner within the base cell when the elastic contrast of two phases $E_2/E_1$ is different. Generally, the reinforcements in the base cells are oriented to efficiently carry the applied load at the macro level.

5.3. 3D example

A 3D cantilever is given here to further demonstrate the effectiveness of the proposed optimization algorithm. Fig. 6 shows the design domain at the macro level, supporting and loading conditions of the 3D cantilever with length $L = 20$, width $B = 10$ and height $H = 20$. Young’s modulus of the phase 2 $E_2$ is assumed to be 1.8.

Table 5 shows the optimized macrostructures and microstructures with different weight fraction constraints. Table 5 indicates that the stiffness of the optimized composite in $y$-direction is

Table 5
The optimized macrostructures and their composite microstructures for the 3D cantilever with various weight constraints.

<table>
<thead>
<tr>
<th>Weight fraction $W_f$ (%)</th>
<th>Objective C</th>
<th>Macro structure</th>
<th>Micro base cell</th>
<th>Micro base cell (phase 1 only)</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>13.01</td>
<td>![Macro structure image]</td>
<td>![Micro base cell image]</td>
<td>![Micro base cell image]</td>
</tr>
<tr>
<td>40</td>
<td>13.33</td>
<td>![Macro structure image]</td>
<td>![Micro base cell image]</td>
<td>![Micro base cell image]</td>
</tr>
<tr>
<td>20</td>
<td>13.68</td>
<td>![Macro structure image]</td>
<td>![Micro base cell image]</td>
<td>![Micro base cell image]</td>
</tr>
<tr>
<td>10</td>
<td>14.07</td>
<td>![Macro structure image]</td>
<td>![Micro base cell image]</td>
<td>![Micro base cell image]</td>
</tr>
</tbody>
</table>
highest and the stiffness in z-direction is lowest except for the case with the weight fraction $W_f^i = 10\%$ where the material is totally composed of phase 2. With the decrease of the weight fraction $W_f^i$ the optimized macrostructure has no significant change but the material microstructure changes significantly until the material is totally composed of phase 2. This phenomenon is very similar to the 2D cases.

6. Conclusions

This paper has developed a two-scale topology optimization algorithm based on the BESO method for the concurrent design of macrostructures and their composite microstructures. Different from the previous multi-scale topology optimization method which uncouples the designs at different scales [23–25], the proposed two-scale optimization approach integrates the design of macrostructures with that of material microstructures and obtains optimized macrostructures and material microstructures simultaneously under a total weight constraint. Material distribution at the two scales can be adjusted automatically to efficiently carry the applied load at the macro level. Numerical examples demonstrate that designs at the macro and micro scales interact strongly with each other. The two-scale optimization can obtain a better design than that from a one-scale optimization since the design degrees of freedom are increased significantly. This study has given the optimized topologies of macrostructures and composite material microstructures with various weight fractions. This study has revealed that the optimized macrostructure has no significant changes when the total weight fraction decreases, but at the same time the stiffer and heavier material (phase 1) within the base cell at the micro scale level becomes less and less until it is totally replaced with the softer and lighter material (phase 2). Further study can extend the methodology to the concurrent design of structure and material under multi-physical fields.

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References